On compatible metrics and diagonalizability of non-locally bi-Hamiltonian systems of hydrodynamic type[1]

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Abstract

We study bi-Hamiltonian systems of hydrodynamic type with non-singular (semisimple) non-local bi-Hamiltonian structures and prove that such systems of hydrodynamic type are diagonalizable. Moreover, we prove that for an arbitrary non-singular (semisimple) non-locally bi-Hamiltonian system of hydrodynamic type, there exist local coordinates (Riemann invariants) such that all the related matrix differential-geometric objects, namely, the matrix $V^i_j(u)$ of this system of hydrodynamic type, the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ and the affinors $(w_{1,n})^i_j(u)$ and $(w_{2,n})^i_j(u)$ of the non-singular non-local bi-Hamiltonian structure of this system, are diagonal in these local coordinates. The proof is a natural consequence of the general results of the theory of compatible metrics and the theory of non-local bi-Hamiltonian structures developed earlier by the present author in [21]–[33].

Introduction

In this paper we consider (1+1)-dimensional non-singular (semisimple) non-locally bi-Hamiltonian systems of hydrodynamic type and prove their diagonalizability. Moreover, we prove that for an arbitrary non-singular (semisimple) non-locally bi-Hamiltonian system of hydrodynamic type, there exist local coordinates (Riemann invariants) such that all the related matrix differential-geometric objects, namely, the matrix $V^i_j(u)$ of this system of hydrodynamic type, the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ and the affinors $(w_{1,n})^i_j(u)$ and $(w_{2,n})^i_j(u)$ of the non-singular non-local bi-Hamiltonian structure of this system, are diagonal in these local coordinates. Let us give here very briefly basic well-known notions and results necessary for us. Recall that $(1 + 1)$-dimensional *systems of hydrodynamic type* [1] are arbitrary $(1 + 1)$-dimensional evolution quasilinear systems of first-order partial differential equations, i.e., equations of the form

$$u^i_t = V^i_j(u)u^j_x, \qquad 1 \le i, j \le N, \tag{1}$$

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where $u = (u^1, \ldots, u^N)$ are local coordinates on a certain smooth N-dimensional manifold or in a domain of $\mathbb{R}^N$ (or $\mathbb{C}^N$); $u^i(x)$ are functions (fields) of one variable x that are evolving with respect to t; $V^i_j(u)$ is an arbitrary $(N \times N)$-matrix depending on u (this matrix is a mixed tensor of the type $(1, 1)$, i.e., an affinor, with respect to local changes of coordinates u).

Non-locally Hamiltonian systems of hydrodynamic type

We will consider systems of the form (1) that are Hamiltonian with respect to arbitrary non-degenerate non-local Poisson brackets of hydrodynamic type (the Ferapontov brackets [2], see also [3] and [1] for the Mokhov–Ferapontov brackets and the Dubrovin–Novikov brackets in partial non-local and local cases, respectively), i.e.,

$$u^i_t = V^i_j(u) u^j_x = \{u^i(x), H\}, \quad 1 \le i, j \le N, \tag{2}$$

where the functional

$$H = \int h(u(x)) dx \tag{3}$$

is the Hamiltonian of the system (2) (the function $h(u)$ is the density of the Hamiltonian) and the Poisson bracket has the form

$$\{u^i(x), u^j(y)\} = P^{ij} \delta(x - y), \quad 1 \le i, j \le N, \tag{4}$$

$$\begin{aligned} P^{ij} &= g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x))\, u^k_x + \\ &+ \sum_{m,n=1}^{L} \mu^{mn} (w_m)^i_k(u(x)) u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_n)^j_s(u(x)) u^s_x, \end{aligned} \tag{5}$$

where the coefficients $g^{ij}(u)$, $b^{ij}_k(u)$, and $(w_n)^i_j(u)$, $1 \le i, j, k \le N$, $1 \le n \le L$, are smooth functions of local coordinates, $\det(g^{ij}(u)) \ne 0$, μ^{mn} is an arbitrary non-degenerate symmetric constant matrix, $\mu^{mn} = \mu^{nm}$, $\mu^{mn} = \text{const}$, $\det(\mu^{mn}) \ne 0$. For two arbitrary functionals I and J the Poisson bracket (4), (5) has the form

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} P^{ij} \frac{\delta J}{\delta u^j(x)} dx. \tag{6}$$

Poisson brackets of the form (5), (6) were introduced and studied by Ferapontov in [2]; these brackets are a non-local generalization of the Dubrovin–Novikov brackets (local Poisson brackets of hydrodynamic type generated by flat metrics $g^{ij}(u)$; there are no non-local terms in this case, $L = 0$, or $(w_n)^i_j(u) = 0$) [1] and the Mokhov–Ferapontov brackets (non-local Poisson brackets of hydrodynamic type generated by metrics of constant curvature K; in this case $L = 1$, $\mu^{11} = K$, $(w_1)^i_j(u) = \delta^i_j$) [3]. Ferapontov proved that a

non-local operator P^{ij} of the form (5) gives a Poisson bracket (6) if and only if there is an N-dimensional submanifold with flat normal bundle in an $(N+L)$-dimensional pseudo-Euclidean space such that $g^{ij}(u)$ is the contravariant first fundamental form; $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$; $\Gamma^j_{sk}(u)$ are the Christoffel symbols of the Levi-Civita connection of the metric $g^{ij}(u)$; $(w_n)^i_j(u)$, $1 \leq n \leq L$, are the Weingarten operators (the Weingarten affinors) of the submanifold; and μ^{mn} is the Gram matrix of the corresponding parallel bases in the normal spaces of the submanifold (all torsion forms of the submanifold with flat normal bundle vanish in these bases in the normal spaces).

In other words, the non-local operator (5) gives a Poisson bracket (6) if and only if its coefficients satisfy the relations (see also [4])

$$g^{ij} = g^{ji}, \tag{7}$$

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k, \tag{8}$$

$$g^{is} b^{jk}_s = g^{js} b^{ik}_s, \tag{9}$$

$$g^{is} (w_n)^j_s = g^{js} (w_n)^i_s, \tag{10}$$

$$(w_n)^i_s (w_m)^s_j = (w_m)^i_s (w_n)^s_j, \tag{11}$$

$$g^{is} g^{jr} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{jr} b^{ik}_s (w_n)^s_r = g^{js} g^{ir} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{ir} b^{jk}_s (w_n)^s_r, \tag{12}$$

$$g^{is} \left(\frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s} \right) + b^{ij}_s b^{sk}_r - b^{ik}_s b^{sj}_r = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} g^{is} \left((w_m)^j_r (w_n)^k_s - (w_m)^j_s (w_n)^k_r \right). \tag{13}$$

Non-locally Hamiltonian affinors

The Hamiltonian H of the system (2)–(6) must also be a first integral of all the systems of hydrodynamic type that are given by the affinors $(w_n)^i_j(u)$, $1 \leq n \leq L$, of the non-local operator (5) (these systems are called the *structural flows* of the non-local Poisson bracket (4)–(6)) [2]:

$$u^i_{t_n} = (w_n)^i_j(u) u^j_x, \quad H_{t_n} = 0, \quad 1 \leq n \leq L. \tag{14}$$

For each n, $1 \leq n \leq L$, there exist a function $f_n(u)$ such that

$$\frac{\partial h}{\partial u^j} (w_n)^j_s(u) = \frac{\partial f_n}{\partial u^s}, \quad 1 \leq n \leq L. \tag{15}$$

In this case the affinor $V^i_j(u)$ of the system of hydrodynamic type (2)–(6) has the form

$$V^i_j(u) = g^{is}(u) \frac{\partial^2 h}{\partial u^s \partial u^j} - g^{is}(u) \Gamma^p_{sj}(u) \frac{\partial h}{\partial u^p} + \sum_{m,n=1}^{L} \mu^{mn} (w_m)^i_j(u) f_n(u), \tag{16}$$

i.e.,

$$V^i_j(u) = g^{is}(u)\nabla_s\nabla_j h(u) + \sum_{m,n=1}^{L} \mu^{mn}(w_m)^i_j(u) f_n(u) =$$

$$= \nabla^i\nabla_j h(u) + \sum_{m,n=1}^{L} \mu^{mn}(w_m)^i_j(u) f_n(u), \tag{17}$$

where ∇_k is the covariant differentiation generated by the Levi-Civita connection $\Gamma^j_{sk}(u)$ of the metric $g^{ij}(u)$.

We will call an affinor $V^i_j(u)$ *Hamiltonian* (or *non-locally Hamiltonian*) if there exist an N-dimensional submanifold with flat normal bundle in an $(N+L)$-dimensional pseudo-Euclidean space and functions $h(u)$ and $f_n(u)$, $1 \leq n \leq L$, such that the affinor $V^i_j(u)$ has the form (17), where $g^{ij}(u)$ is the contravariant first fundamental form of the submanifold; $\Gamma^j_{sk}(u)$ are the Christoffel symbols of the Levi-Civita connection of the metric $g^{ij}(u)$; $(w_n)^i_j(u)$, $1 \leq n \leq L$, are the Weingarten operators of the submanifold; and μ^{mn} is the Gram matrix of the corresponding parallel bases in the normal spaces of the submanifold (such that all torsion forms of the submanifold with flat normal bundle vanish in these bases in the normal spaces), and the functions $h(u)$ and $f_n(u)$, $1 \leq n \leq L$, satisfy relations (15):

$$(w_n)^j_s(u)\nabla_j h(u) = \nabla_s f_n(u), \quad 1 \leq n \leq L. \tag{18}$$

In this case we will also speak that the affinor $V^i_j(u)$ (17), (18) is Hamiltonian with respect to the corresponding non-local Poisson bracket of hydrodynamic type (4)–(6). Obviously, this definition is invariant. We note that one can also consider it as a definition of a non-locally Hamiltonian system of hydrodynamic type (1).

Affinors that are Hamiltonian with respect to the Dubrovin–Novikov brackets (*locally Hamiltonian affinors*) were studied in detail by Tsarev in the remarkable work [5]. Affinors that are Hamiltonian with respect to the non-local Mokhov–Ferapontov brackets (the Mokhov–Ferapontov affinors) were studied in detail in the paper [3].

Using relations (7)–(13) it is easy to prove that the following relations always hold for non-locally Hamiltonian affinors $V^i_j(u)$:

$$g_{is}(u)V^s_j(u) = g_{js}(u)V^s_i(u), \tag{19}$$

$$\nabla_j V^i_k(u) = \nabla_k V^i_j(u). \tag{20}$$

Here $g_{ij}(u)$ is the inverse of the matrix $g^{ij}(u)$, $g_{is}(u)g^{sj}(u) = \delta^j_i$ (the covariant metric).

For locally Hamiltonian affinors these important relations are very simple in flat local coordinates of the metric $g_{ij}(u)$ and Tsarev proved that in this flat case relations (19) and (20) are not only necessary but also sufficient for an affinor to be locally Hamiltonian

(an affinor $V^i_j(u)$ is locally Hamiltonian if and only if there exists a flat metric $g_{ij}(u)$ such that relations (19) and (20) hold) [5]. This result was generalized to the case of the Mokhov–Ferapontov affinors in [3]: an affinor $V^i_j(u)$ is Hamiltonian with respect to a non-local Mokhov–Ferapontov bracket if and only if there exists a metric $g_{ij}(u)$ of constant curvature such that relations (19) and (20) hold.

Let the affinor $V^i_j(u)$ of a system of hydrodynamic type (1) satisfy relations (19) and (20), i.e., there exists a metric $g_{ij}(u)$ such that relations (19) and (20) hold. If this system of hydrodynamic type is diagonalizable, i.e., there exist local coordinates such that the affinor $V^i_j(u)$ is a diagonal matrix $V^i_j(u) = V^i(u)\delta^i_j$ in these special local coordinates (such local coordinates are called *Riemann invariants*), and strictly hyperbolic, i.e., all the eigenvalues $V^i(u)$, $1 \leq i \leq N$, are distinct ($V^i(u) \neq V^j(u)$ when $i \neq j$), then it can be integrated by the generalized hodograph method (Tsarev, see [5]). In this case, relation (19) is equivalent to the condition that the metric $g_{ij}(u)$ is also diagonal in these special local coordinates, $g_{ij}(u) = g_i(u)\delta_{ij}$, i.e., the Riemann invariants are orthogonal curvilinear coordinates in the corresponding pseudo-Riemannian space, and relation (20) is equivalent to the condition

$$\frac{\partial V^i}{\partial u^j} = \frac{\partial \ln \sqrt{g_i(u)}}{\partial u^j}(V^j(u) - V^i(u)), \quad i \neq j. \tag{21}$$

Hence, the following relation holds for the eigenvalues $V^i(u)$:

$$\frac{\partial}{\partial u^k}\left(\frac{1}{(V^j(u) - V^i(u))}\frac{\partial V^i}{\partial u^j}\right) = \frac{\partial}{\partial u^j}\left(\frac{1}{(V^k(u) - V^i(u))}\frac{\partial V^i}{\partial u^k}\right). \tag{22}$$

A strictly hyperbolic diagonal system of hydrodynamic type is called *semi-Hamiltonian* if relations (22) hold (Tsarev, see [5]). In [5] Tsarev proved that any strictly hyperbolic diagonal semi-Hamiltonian system of hydrodynamic type is integrable by the generalized hodograph method.

Diagonalizable affinors

Recall that the very important problem of diagonalizability for an affinor, which had been posed, in fact, by Riemann, was completely solved by Haantjes in [6] on the base of the previous Nijenhuis' results [7]. An affinor $V^i_j(u)$ is diagonalizable by a local change of coordinates in a domain if and only if it is diagonalizable at any point and its Haantjes tensor vanishes. The Haantjes tensor of an affinor $V(u) = V^i_j(u)$ is the following tensor of the type $(1,2)$ (a skew-symmetric vector-valued 2-form) generated by the affinor $V^i_j(u)$:

$$H(X,Y) = N(V(X),V(Y)) + V^2(N(X,Y)) - V(N(X,V(Y))) - V(N(V(X),Y)), \tag{23}$$

where $X(u)$ and $Y(u)$ are arbitrary vector fields, $V(X)$ is the vector field $V^i_j(u)X^j(u)$, $N(X,Y)$ is the Nijenhuis tensor of the affinor $V^i_j(u)$, i.e., the following tensor of the type $(1,2)$ (a skew-symmetric vector-valued 2-form) generated by the affinor $V^i_j(u)$:

$$N(X,Y) = [V(X),V(Y)] + V^2([X,Y]) - V([X,V(Y)]) - V([V(X),Y]), \tag{24}$$

where $[X,Y]$ is the commutator of the vector fields $X(u)$ and $Y(u)$. In components, the Nijenhuis tensor of the affinor $V^i_j(u)$ has the form

$$N^k_{ij}(u) = V^s_i(u)\frac{\partial V^k_j}{\partial u^s} - V^s_j(u)\frac{\partial V^k_i}{\partial u^s} + V^k_s(u)\frac{\partial V^s_i}{\partial u^j} - V^k_s(u)\frac{\partial V^s_j}{\partial u^i} \tag{25}$$

and the Haantjes tensor of the affinor $V^i_j(u)$ has the form

$$\begin{aligned} H^i_{jk}(u) = V^i_s(u)V^s_r(u)N^r_{jk}(u) - V^i_s(u)N^s_{rk}(u)V^r_j(u) - \\ -V^i_s(u)N^s_{jr}(u)V^r_k(u) + N^i_{sr}(u)V^s_j(u)V^r_k(u). \end{aligned} \tag{26}$$

Recall also that invariant tensor conditions that a strictly hyperbolic system of hydrodynamic type is semi-Hamiltonian were found in [8].

Non-locally bi-Hamiltonian systems of hydrodynamic type

We will consider bi-Hamiltonian systems of hydrodynamic type. Recall that two Poisson brackets are called *compatible* if any linear combination of these Poisson brackets is also a Poisson bracket [9], and a system of equations that is Hamiltonian with respect to two linearly independent compatible Poisson brackets is called *bi-Hamiltonian*. In this paper we will consider systems of hydrodynamic type that are bi-Hamiltonian with respect to two linearly independent compatible non-degenerate non-local Poisson brackets of hydrodynamic type (4)–(6),

$$u^i_t = V^i_j(u)u^j_x = \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2, \quad 1 \le i,j \le N, \tag{27}$$

$$H_1 = \int h_1(u(x))dx, \quad H_2 = \int h_2(u(x))dx, \tag{28}$$

$$\{I,J\}_1 = \int \frac{\delta I}{\delta u^i(x)} P^{ij}_1 \frac{\delta J}{\delta u^j(x)} dx, \quad \{I,J\}_2 = \int \frac{\delta I}{\delta u^i(x)} P^{ij}_2 \frac{\delta J}{\delta u^j(x)} dx, \tag{29}$$

$$\begin{aligned} P^{ij}_1 = g^{ij}_1(u(x))\frac{d}{dx} + b^{ij}_{1,k}(u(x))\, u^k_x + \\ + \sum_{m,n=1}^{L} \mu^{mn}_1 (w_{1,m})^i_k(u(x))u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_{1,n})^j_s(u(x))u^s_x, \end{aligned} \tag{30}$$

$$P_2^{ij} = g_2^{ij}(u(x))\frac{d}{dx} + b_{2,k}^{ij}(u(x))\, u_x^k +$$
$$+ \sum_{m,n=1}^{L} \mu_2^{mn} (w_{2,m})_k^i(u(x)) u_x^k \left(\frac{d}{dx}\right)^{-1} \circ (w_{2,n})_s^j(u(x)) u_x^s. \qquad (31)$$

We will call an affinor $V_j^i(u)$ *bi-Hamiltonian* (or *non-locally bi-Hamiltonian*) if this affinor is Hamiltonian with respect to two linearly independent compatible non-degenerate non-local Poisson brackets of hydrodynamic type (4)–(6). This definition is invariant.

In this paper we prove that (1+1)-dimensional non-singular (semisimple) non-locally bi-Hamiltonian systems of hydrodynamic type are diagonalizable. Recall that a pair of pseudo-Riemannian metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is called *non-singular* (or *semisimple*) if the eigenvalues of this pair of metrics, i.e., the roots of the equation

$$\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0, \qquad (32)$$

are distinct. In this case the non-locally bi-Hamiltonian system of hydrodynamic type (27)–(31) and the corresponding bi-Hamiltonian affinor are also called *non-singular* (or *semisimple*).

It is important to note that, generally speaking, integrable bi-Hamiltonian systems of hydrodynamic type are not necessarily diagonalizable if we consider an other class of compatible Poisson brackets (even if both the compatible Poisson brackets are local). This is a nontrivial fact and we give here a very important example in detail.

Example (*a non-diagonalizable integrable bi-Hamiltonian system of hydrodynamic type* [10]-[12]). Let us consider the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde equations, see [13]–[16]) for a function (a *potential*) $\Phi = \Phi(u^1, \ldots, u^N)$,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \qquad (33)$$

where η^{ij} is an arbitrary constant nondegenerate symmetric matrix, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} =$ const, $\det(\eta^{ij}) \neq 0$. We recall that the associativity equations (33) are consistent and integrable by the inverse scattering method, they possess a rich set of nontrivial solutions, and each solution $\Phi(u^1, \ldots, u^N)$ of the associativity equations (33) gives N-parameter deformations of special Frobenius algebras (some special commutative associative algebras equipped with nondegenerate invariant symmetric bilinear forms) (see [13]). Indeed, consider algebras $A(u)$ in an N-dimensional vector space with the basis $e_1, \ldots, e_N$ and the multiplication (see [13])

$$e_i \circ e_j = c_{ij}^k(u) e_k, \qquad c_{ij}^k(u) = \eta^{ks} \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}. \qquad (34)$$

For all values of the parameters $u = (u^1, \dots, u^N)$ the algebras $A(u)$ are commutative, $e_i \circ e_j = e_j \circ e_i$, and the associativity condition

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \tag{35}$$

in the algebras $A(u)$ is equivalent to equations (33). The matrix η_{ij} inverse to the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, defines a nondegenerate invariant symmetric bilinear form on the algebras $A(u)$,

$$\langle e_i, e_j \rangle = \eta_{ij}, \qquad \langle e_i \circ e_j, e_k \rangle = \langle e_i, e_j \circ e_k \rangle. \tag{36}$$

Recall that locally the tangent space at every point of any Frobenius manifold (see [13]) possesses the structure of Frobenius algebra (34)–(36), which is determined by a solution of the associativity equations (33) and smoothly depends on the point. Let $N = 3$ and the metric η_{ij} be antidiagonal

$$(\eta_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{37}$$

and the function $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}(u^1)^2 u^3 + \frac{1}{2} u^1 (u^2)^2 + f(u^2, u^3).$$

In this case e_1 is the unit in the Frobenius algebra (34)–(36), and the associativity equations (33) for the function $\Phi(u)$ are equivalent to the following remarkable integrable Dubrovin equation for the function $f(u^2, u^3)$:

$$\frac{\partial^3 f}{\partial (u^3)^3} = \left(\frac{\partial^3 f}{\partial (u^2)^2 \partial u^3} \right)^2 - \frac{\partial^3 f}{\partial (u^2)^3} \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{38}$$

We introduce here new independent variables: $x = u^2$, $t = u^3$. The equation (38) takes the form

$$f_{ttt} = (f_{xxt})^2 - f_{xxx} f_{xtt}. \tag{39}$$

This equation is connected to quantum cohomology of projective plane and classical problems of enumerative geometry (see [17]).

It was proved by the present author in [18] (see also [19], [10]–[12]) that the equation (39) is equivalent to the integrable non-diagonalizable system of hydrodynamic type

$$\begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a^3 & 2a^2 & -a^1 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}_x, \tag{40}$$

$$a^1 = f_{xxx}, \quad a^2 = f_{xxt}, \quad a^3 = f_{xtt}. \tag{41}$$

The first Hamiltonian structure of system (40) given by a Dubrovin–Novikov bracket was found in [20]:

$$\{I, J\}_1 = \int \frac{\delta I}{\delta a^i(x)} M_1^{ij} \frac{\delta J}{\delta a^j(x)} dx, \tag{42}$$

$$M_1 = (M_1^{ij}) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2}a^1 & a^2 \\ \frac{1}{2}a^1 & a^2 & \frac{3}{2}a^3 \\ a^2 & \frac{3}{2}a^3 & 2((a^2)^2 - a^1a^3) \end{pmatrix} \frac{d}{dx} +$$
$$+ \begin{pmatrix} 0 & \frac{1}{2}a^1_x & a^2_x \\ 0 & \frac{1}{2}a^2_x & a^3_x \\ 0 & \frac{1}{2}a^3_x & ((a^2)^2 - a^1a^3)_x \end{pmatrix}. \tag{43}$$

The metric

$$(g_1^{ij}(a)) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2}a^1 & a^2 \\ \frac{1}{2}a^1 & a^2 & \frac{3}{2}a^3 \\ a^2 & \frac{3}{2}a^3 & 2((a^2)^2 - a^1a^3) \end{pmatrix} \tag{44}$$

is flat and the Poisson bracket of hydrodynamic type (42), (43) is local (a Dubrovin–Novikov bracket). The functional

$$H_1 = \int a^3 dx \tag{45}$$

is the corresponding Hamiltonian of system (40).

The bi-Hamiltonian structure of system (40) was found in [10] (see also [11], [12]). The second Hamiltonian structure of system (40) is given by a homogeneous third-order Dubrovin–Novikov bracket:

$$\{I, J\}_2 = \int \frac{\delta I}{\delta a^i(x)} M_2^{ij} \frac{\delta J}{\delta a^j(x)} dx, \tag{46}$$

$$M_2 = (M_2^{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a^1 \\ 1 & -a^1 & (a^1)^2 + 2a^2 \end{pmatrix} \left(\frac{d}{dx}\right)^3 +$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2a^1_x \\ 0 & -a^1_x & 3(a^2_x + a^1a^1_x) \end{pmatrix} \left(\frac{d}{dx}\right)^2 +$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2_{xx} + (a^1_x)^2 + a^1a^1_{xx} \end{pmatrix} \frac{d}{dx}. \tag{47}$$

The second Poisson bracket (46), (47) is compatible with the first Poisson bracket (42), (43).

The metric

$$(g_1^{ij}(a)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a^1 \\ 1 & -a^1 & (a^1)^2 + 2a^2 \end{pmatrix} \tag{48}$$

is flat. The non-local functional

$$H_2 = -\int \left(\frac{1}{2} a^1 \left(\left(\frac{d}{dx} \right)^{-1} a^2 \right)^2 + \left(\left(\frac{d}{dx} \right)^{-1} a^2 \right) \left(\left(\frac{d}{dx} \right)^{-1} a^3 \right) \right) dx \tag{49}$$

is the corresponding Hamiltonian of system (40).

First of all, we note that the class of non-locally bi-Hamiltonian systems of hydrodynamic type (27)–(31) is very rich, there are many well-known important examples arising in various applications. An explicit general construction of locally and non-locally bi-Hamiltonian systems of hydrodynamic type and corresponding integrable hierarchies that are generated by pairs of compatible Poisson brackets of hydrodynamic type and integrable description of local and non-local compatible Poisson brackets of hydrodynamic type were found and studied by the present author in [21]–[33], [4] (see also [34], [35], [13]).

Compatible metrics and non-locally bi-Hamiltonian systems of hydrodynamic type

Now recall some necessary basic facts of the general theory of compatible metrics [27]–[32]. Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called *compatible* if for any linear combination of these metrics

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u), \tag{50}$$

where λ_1 and λ_2 are arbitrary constants such that $\det(g^{ij}(u)) \neq 0$, the coefficients of the corresponding Levi–Civita connections and the components of the corresponding Riemannian curvature tensors are related by the same linear formula [27]–[29]:

$$\Gamma_k^{ij}(u) = \lambda_1 \Gamma_{1,k}^{ij}(u) + \lambda_2 \Gamma_{2,k}^{ij}(u), \tag{51}$$

$$R_{kl}^{ij}(u) = \lambda_1 R_{1,kl}^{ij}(u) + \lambda_2 R_{2,kl}^{ij}(u). \tag{52}$$

The indices of the coefficients of the Levi–Civita connections $\Gamma_{jk}^i(u)$ and the indices of the Riemannian curvature tensors $R_{jkl}^i(u)$ are raised and lowered by the metrics corresponding to them:

$$\Gamma_k^{ij}(u) = g^{is}(u)\Gamma_{sk}^j(u), \quad \Gamma_{jk}^i(u) = \frac{1}{2} g^{is}(u) \left(\frac{\partial g_{sk}}{\partial u^j} + \frac{\partial g_{js}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^s} \right),$$

$$R_{kl}^{ij}(u) = g^{is}(u) R_{skl}^j(u),$$

$$R_{jkl}^i(u) = \frac{\partial \Gamma_{jl}^i}{\partial u^k} - \frac{\partial \Gamma_{jk}^i}{\partial u^l} + \Gamma_{pk}^i(u)\Gamma_{jl}^p(u) - \Gamma_{pl}^i(u)\Gamma_{jk}^p(u).$$

Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called *almost compatible* if for any linear combination of these metrics (50) relation (51) holds [27]–[29].

Let us introduce the affinor

$$v_j^i(u) = g_1^{is}(u) g_{2,sj}(u) \tag{53}$$

and consider the Nijenhuis tensor of this affinor

$$N_{ij}^k(u) = v_i^s(u) \frac{\partial v_j^k}{\partial u^s} - v_j^s(u) \frac{\partial v_i^k}{\partial u^s} + v_s^k(u) \frac{\partial v_i^s}{\partial u^j} - v_s^k(u) \frac{\partial v_j^s}{\partial u^i}. \tag{54}$$

Theorem 1 [27]–[29]. *Any two metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are almost compatible if and only if the corresponding Nijenhuis tensor $N_{ij}^k(u)$ (54) vanishes.*

Assume that a pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is non-singular, i.e., the eigenvalues of this pair of metrics are distinct. Furthermore, assume that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are almost compatible, i.e., the corresponding Nijenhuis tensor $N_{ij}^k(u)$ (54) vanishes. It was proved in our papers [27]–[29] that, in this case, the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible, i.e., relation (52) holds.

It is obvious that the eigenvalues of the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ coincide with the eigenvalues of the affinor $v_j^i(u)$ (53). But it is well known that if all eigenvalues of an affinor are distinct, then it always follows from the vanishing of the Nijenhuis tensor of this affinor that there exist special local coordinates (*Riemann invariants*) such that, in these coordinates, the affinor reduces to a diagonal form in the corresponding neighbourhood [7] (see also [6]).

Hence, we can consider that the affinor $v_j^i(u)$ is diagonal in the local coordinates (Riemann invariants) $u^1, ..., u^N$, i.e.,

$$v_j^i(u) = f^i(u) \delta_j^i, \tag{55}$$

where is no summation over the index i. By our assumption, the eigenvalues $f^i(u)$, $i = 1, ..., N$, coinciding with the eigenvalues of the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are distinct:

$$f^i(u) \neq f^j(u) \quad \text{if} \quad i \neq j. \tag{56}$$

Lemma 1. *If the affinor $v_j^i(u)$ (53) is diagonal in certain local coordinates (Riemann invariants) and all its eigenvalues are distinct, then, in these coordinates, the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are also necessarily diagonal, i.e., in this case both the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are diagonal in the Riemann invariants.*

Actually, we have

$$g_1^{ij}(u) = f^i(u) g_2^{ij}(u).$$

It follows from the symmetry of the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ that for any indices i and j

$$(f^i(u) - f^j(u))g_2^{ij}(u) = 0, \tag{57}$$

where is no summation over indices, i.e.,

$$g_2^{ij}(u) = g_1^{ij}(u) = 0 \quad \text{if} \quad i \neq j.$$

Lemma 2. *Let an affinor $w_j^i(u)$ be diagonal in certain local coordinates (Riemann invariants) $u = (u^1, ..., u^N)$, i.e., $w_j^i(u) = \mu^i(u)\delta_j^i$.*

1) If all the eigenvalues $\mu^i(u)$, $i = 1, ..., N$, of the diagonal affinor are distinct, i.e., $\mu^i(u) \neq \mu^j(u)$ for $i \neq j$, then the Nijenhuis tensor of this affinor vanishes if and only if the ith eigenvalue $\mu^i(u)$ depends only on the coordinate u^i.

2) If all the eigenvalues coincide, then the Nijenhuis tensor vanishes.

3) In the general case of an arbitrary diagonal affinor $w_j^i(u) = \mu^i(u)\delta_j^i$, the Nijenhuis tensor vanishes if and only if

$$\frac{\partial \mu^i}{\partial u^j} = 0 \tag{58}$$

for all indices i and j such that $\mu^i(u) \neq \mu^j(u)$.

It follows from Lemmas 1 and 2 that for any non-singular pair of almost compatible metrics there always exist local coordinates (Riemann invariants) in which the metrics have the form

$$g_2^{ij}(u) = g^i(u)\delta^{ij}, \quad g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}.$$

Moreover, any pair of diagonal metrics of the form $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$ for any nonzero functions $f^i(u^i)$, $i = 1, ..., N$, (here they can be, for example, coinciding nonzero constants, i.e., the pair of metrics may be "singular") is almost compatible, since the corresponding Nijenhuis tensor always vanishes for any pair of metrics of this form. It was proved in our papers [28], [29] that an arbitrary pair of diagonal metrics of such the form, $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, for arbitrary nonzero functions $f^i(u^i)$, $i = 1, ..., N$, (the pair of metrics may be "singular"), is always compatible, i.e., in this case, relation (52) holds. We note that, as it was shown in [27]–[29], in general almost compatible metrics are not necessarily compatible even in the case of flat metrics or metrics of constant curvature, i.e., in the case of the Dubrovin–Novikov or the Mokhov–Ferapontov brackets, but if a pair of almost compatible metrics is not compatible, then this pair of metrics must be singular. Thus, we have the following important statements.

Theorem 2 [27]–[29]. *If a pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is non-singular, i.e., the roots of the equation*

$$\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0 \tag{59}$$

are distinct, then it follows from the vanishing of the Nijenhuis tensor of the affinor $v^i_j(u) = g^{is}_1(u) g_{2,sj}(u)$ *that the metrics* $g^{ij}_1(u)$ *and* $g^{ij}_2(u)$ *are compatible. Thus, a non-singular pair of metrics is compatible if and only if the metrics are almost compatible.*

Theorem 3 [27]–[29]. *An arbitrary non-singular pair of metrics is compatible if and only if there exist local coordinates (Riemann invariants)* $u = (u^1, ..., u^N)$ *such that both the metrics are diagonal in these coordinates and have the following special form:* $g^{ij}_2(u) = g^i(u)\delta^{ij}$ *and* $g^{ij}_1(u) = f^i(u^i) g^i(u) \delta^{ij}$*, where one of the metrics, here* $g^{ij}_2(u)$*, is an arbitrary diagonal metric and* $f^i(u^i)$*,* $i = 1, ..., N$*, are arbitrary (generally speaking, complex) nonzero functions of single variables. If some of the functions* $f^i(u^i)$*,* $i = 1, ..., N$*, are coinciding nonzero constants, then the pair of metrics of this form is singular but, nevertheless, compatible.*

Theorem 4 [28]. *If non-local Poisson brackets of hydrodynamic type* (29)–(31) *are compatible, then their metrics are compatible.*

In [2] Ferapontov proved that a bracket (4)–(6) is a Poisson bracket, i.e., it is skew-symmetric and satisfies the Jacobi identity, if and only if

(1) $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi–Civita connection),

(2) the pseudo-Riemannian metric $g^{ij}(u)$ and the set of affinors $(w_n)^i_j(u)$ satisfy the relations:

$$g_{ik}(u)(w_n)^k_j(u) = g_{jk}(u)(w_n)^k_i(u), \quad n = 1, ..., L, \tag{60}$$

$$\nabla_k (w_n)^i_j(u) = \nabla_j (w_n)^i_k(u), \quad n = 1, ..., L, \tag{61}$$

$$R^{ij}_{kl}(u) = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \left((w_m)^i_l(u)(w_n)^j_k(u) - (w_m)^j_l(u)(w_n)^i_k(u) \right). \tag{62}$$

Moreover, the family of affinors $w_n(u)$ is commutative: $[w_m, w_n] = 0$.

If non-local Poisson brackets of hydrodynamic type (29)–(31) are compatible, then it follows from the conditions of compatibility and from Ferapontov's theorem that, first, relation (51) holds, i.e., the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ are almost compatible, and, secondly, the curvature tensor for the metric $g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u)$ has the form

$$\begin{aligned} R^{ij}_{kl}(u) = &\sum_{m=1}^{L_1} \sum_{n=1}^{L_1} \lambda_1 \mu^{mn}_1 \left((w_{1,m})^i_l(u)(w_{1,n})^j_k(u) - (w_{1,m})^j_l(u)(w_{1,n})^i_k(u) \right) + \\ &+ \sum_{m=1}^{L_2} \sum_{n=1}^{L_2} \lambda_2 \mu^{mn}_2 \left((w_{2,m})^i_l(u)(w_{2,n})^j_k(u) - (w_{2,m})^j_l(u)(w_{2,n})^i_k(u) \right) = \\ &= \lambda_1 R^{ij}_{1,kl}(u) + \lambda_2 R^{ij}_{2,kl}(u), \end{aligned}$$

i.e., relation (52) holds and hence the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible.

Theorem 5 [28]. *Let two non-local Poisson brackets of hydrodynamic type* (29)–(31) *correspond to submanifolds with holonomic net of curvature lines and be given in coordinates of curvature lines. In this case, if the corresponding pair of metrics is non-singular, then the non-local Poisson brackets of hydrodynamic type are compatible if and only if their metrics are compatible.*

In this case the metrics $g_1^{ij}(u) = g_1^i(u)\delta^{ij}$ and $g_2^{ij}(u) = g_2^i(u)\delta^{ij}$, and also the Weingarten operators $(w_{1,n})^i_j(u) = (w_{1,n})^i(u)\delta^i_j$ and $(w_{2,n})^i_j(u) = (w_{2,n})^i(u)\delta^i_j$ are diagonal in the coordinates under consideration. For any such "diagonal" case, condition (60) is automatically fulfilled, all the Weingarten operators commute, conditions (61) and (62) have the following form, respectively:

$$2g^i(u)\frac{\partial (w_n)^i}{\partial u^k} = ((w_n)^i - (w_n)^k)\frac{\partial g^i}{\partial u^k} \qquad \text{for all} \quad i \neq k, \tag{63}$$

$$\begin{aligned} R^{ij}_{j\,i}(u) &= \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}(w_m)^i(u)(w_n)^j(u), \\ R^{ij}_{kl}(u) &= 0 \quad \text{if} \quad i \neq k,\ i \neq l, \quad \text{or if} \quad j \neq k,\ j \neq l. \end{aligned} \tag{64}$$

It follows from non-singularity of the pair of the metrics and from compatibility of the metrics that the corresponding Nijenhuis tensor vanishes and there exist functions $f^i(u^i), i = 1, ..., N$, such that:

$$g_1^i(u) = f^i(u^i)g_2^i(u).$$

Using relations (63) and (64), it is easy to prove that in this case it follows from compatibility of the metrics that an arbitrary linear combination of non-local Poisson brackets under consideration is also a Poisson bracket.

Theorem 5 [33]. *If the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is non-singular, then the non-local Poisson brackets of hydrodynamic type $\{I, J\}_1$ and $\{I, J\}_2$* (29)–(31) *are compatible if and only if the metrics are compatible and both the metrics $g_1^{ij}(u)$, $g_2^{ij}(u)$ and the affinors $(w_{1,n})^i_j(u)$, $(w_{2,n})^i_j(u)$ can be samultaneously diagonalized in a domain of local coordinates.*

It is sufficient to prove here that if the pair of metrics is non-singular and the Poisson brackets are compatible, then both the metrics $g_1^{ij}(u)$, $g_2^{ij}(u)$ and the affinors $(w_{1,n})^i_j(u)$, $(w_{2,n})^i_j(u)$ can be samultaneously diagonalized in a domain of local coordinates. All the rest was already proved above. First of all, it was proved that in this case the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible. Since the pair of metrics is non-singular, there exist local coordinates such that the metrics are diagonal and have the following special form in these coordinates: $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $1 \leq i \leq N$, are functions of single variable. The functions $f^i(u^i)$ are the eigenvalues of

the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$, therefore they are distinct by assumption of the theorem even in the case if they are constants (they can not be coinciding constants). It follows from the compatibility of the Poisson brackets $\{I, J\}_1$ and $\{I, J\}_2$ (it is necessary to consider relation (10) for the pencil $\{I, J\}_1 + \lambda\{I, J\}_2$) that

$$g_1^{is}(w_{2,n})_s^j = g_1^{js}(w_{2,n})_s^i, \tag{65}$$

$$g_2^{is}(w_{1,n})_s^j = g_2^{js}(w_{1,n})_s^i. \tag{66}$$

Besides, from relation (10) for the Poisson brackets $\{I, J\}_1$ and $\{I, J\}_2$ we have

$$g_1^{is}(w_{1,n})_s^j = g_1^{js}(w_{1,n})_s^i, \tag{67}$$

$$g_2^{is}(w_{2,n})_s^j = g_2^{js}(w_{2,n})_s^i. \tag{68}$$

From (65) and (68) in our special local coordinates we obtain

$$g^i(w_{2,n})_i^j = g^j(w_{2,n})_j^i, \tag{69}$$

$$f^i(u^i)g^i(w_{2,n})_i^j = f^j(u^j)g^j(w_{2,n})_j^i. \tag{70}$$

Therefore

$$(w_{2,n})_j^i = \frac{g^i}{g^j}(w_{2,n})_i^j = \frac{f^i(u^i)g^i}{f^j(u^j)g^j}(w_{2,n})_i^j, \tag{71}$$

i.e.,

$$\left(1 - \frac{f^i(u^i)}{f^j(u^j)}\right)(w_{2,n})_i^j = 0. \tag{72}$$

Consequently, since all the functions $f^i(u^i)$ are distinct, we get

$$(w_{2,n})_i^j = 0 \quad \text{for } i \neq j. \tag{73}$$

Similarly, from (66) and (67) we have

$$(w_{1,n})_i^j = 0 \quad \text{for } i \neq j. \tag{74}$$

Thus, both the metrics $g_1^{ij}(u)$, $g_2^{ij}(u)$ and the affinors $(w_{1,n})_j^i(u)$, $(w_{2,n})_j^i(u)$ are diagonal in our special local coordinates.

Now we can prove the main theorem of the paper.

Theorem 6. *For an arbitrary non-singular* (*semisimple*) *non-locally bi-Hamiltonian system of hydrodynamic type* (27)–(31), *there exist local coordinates* (*Riemann invariants*) *such that all the related matrix differential-geometric objects, namely, the matrix* $V_j^i(u)$ *of this system of hydrodynamic type, the metrics* $g_1^{ij}(u)$ *and* $g_2^{ij}(u)$ *and the affinors* $(w_{1,n})_j^i(u)$ *and* $(w_{2,n})_j^i(u)$ *of the non-local bi-Hamiltonian structure of this system, are diagonal in these local coordinates.*

If we have a non-singular (semisimple) non-locally bi-Hamiltonian system of hydrodynamic type (27)–(31), then it was proved above that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ of the non-local bi-Hamiltonian structure of this system are compatible and there exist local coordinates such that $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $i = 1, ..., N$, are distinct nonzero functions of single variable (generally speaking, complex), $f^i(u^i) \neq f^j(u^j)$, $i \neq j$. It was also proved above that the affinors $(w_{1,n})^i_j(u)$ and $(w_{2,n})^i_j(u)$ of the non-local bi-Hamiltonian structure of this system, are diagonal in these local coordinates. Let us prove that the matrix $V^i_j(u)$ of this system is also diagonal in these special local coordinates.

Indeed, in these local coordinates, we have from relations (19):

$$g_i(u)V^i_j(u) = g_j(u)V^j_i(u), \qquad f_i(u^i)g_i(u)V^i_j(u) = f_j(u^j)g_j(u)V^j_i(u). \tag{75}$$

Hence,

$$V^i_j(u) = \frac{g_j(u)}{g_i(u)}V^j_i(u) = \frac{f_j(u^j)g_j(u)}{f_i(u^i)g_i(u)}V^j_i(u), \tag{76}$$

i.e.,

$$\frac{g_j(u)}{g_i(u)}V^j_i(u) = \frac{f_j(u^j)g_j(u)}{f_i(u^i)g_i(u)}V^j_i(u). \tag{77}$$

Thus,

$$(f_i(u^i) - f_j(u^j))V^j_i(u) = 0, \tag{78}$$

i.e.,

$$V^i_j(u) = 0, \quad i \neq j, \tag{79}$$

and the diagonalizability of an arbitrary non-singular (semisimple) non-locally bi-Hamiltonian system of hydrodynamic type is proved:

$$V^i_j(u) = V^i(u)\delta^i_j. \tag{80}$$

The diagonalizability of non-singular (semisimple) locally bi-Hamiltonian systems of hydrodynamic type (27)–(31) was noticed in [35]; it follows immediately from the theory of non-singular pairs of compatible flat metrics [27]–[34].

We note that it does not follow from the proof that $V^i(u) \neq V^j(u)$ if $i \neq j$, i.e., an arbitrary non-singular (semisimple) non-locally bi-Hamiltonian system of hydrodynamic type must not be necessarily strictly hyperbolic but we do not know examples of such systems with some coinciding eigenvalues (velocities) $V^i(u)$. We conjecture that there exist such systems and this is a very interesting problem to find non-singular (semisimple) non-locally bi-Hamiltonian systems of hydrodynamic type that are not strictly hyperbolic (i.e., they have some coinciding eigenvalues $V^i(u)$).

We also note that the non-singularity condition of the pair of metrics is very essential and we conjecture that there exist non-diagonalizable singular non-locally bi-Hamiltonian systems of hydrodynamic type. It is also an interesting problem to find

examples of non-diagonalizable singular non-locally bi-Hamiltonian systems of hydrodynamic type.

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